**Trapping analysis of a magnetic electron by a circularly polarized electromagnetic wave in static electric field**

**Abstract**

An ODE math model is presented to illustrate the trapping effect of electrons in a static electric field by an electromagnetic wave through Normal Doppler Resonance (NDR) and Anomalous Doppler Resonance (ADR) under a uniform background magnetic field. When the electromagnetic wave intensity surpasses a threshold, the electron’s parallel velocity becomes trapped and oscillates within a pseudo-potential well. In this trapping region, energy from the static electric field is continuously converted into gyrokinetic energy. The energy transfer ratio from the static electric field to the gyrokinetic is calculated and compared with predictions from quantum theory, showing good agreement.

**I. Introduction**

The manipulation of particles via electromagnetic (E.M) waves is a powerful and versatile technique, primarily employed for two key objectives: particle acceleration and trapping. In the context of acceleration, one prominent mechanism is autoresonance, wherein a charged particle sustains phase synchronization with an E.M wave through nonlinear effects. A well-known example is the Gyro-Resonant Accelerator[[1-3](#_ENREF_1)], in which electrons resonate within a time-varying magnetic field while interacting with a fixed-frequency E.M wave, the smooth augmentation of the magnetic field is automatically accompanied by a correspondent growth of the relativistic electron mass. Additional acceleration mechanisms include betatron resonance[[4-6](#_ENREF_4)] and ponderomotive-force-driven acceleration[[7](#_ENREF_7), [8](#_ENREF_8)]. On the other hand, Velocity-accelerating can also be achieved through Landau resonant, wherein electrons with velocities near the phase velocity of a longitudinal wave become confined within the associated potential well[[9](#_ENREF_9)], and then through increasing the phase velocity along the electron trajectory, the electron will also be accelerated due to the trapping effect. For particle trapping, spatial confinement can be achieved using methods such as optical tweezers or ponderomotive potential wells generated by the spatial interference patterns of laser beats [[10](#_ENREF_10)].

Besides the physics application, Ream of papers describe the phenomenon of the trapping effect between E.M wave and electron in universal space and plasma device such as tokamak. For example, the trapped electron in chorus wave nonlinearity or time domain structures (TDS)[[11](#_ENREF_11)], the relativistic electron precipitation by EMIC waves (electromagnetic ion cyclotron mode[[12](#_ENREF_12)]), and the resonance with whistle mode wave[[13](#_ENREF_13)].

Although the interaction between test particles and E.M waves has been extensively studied[[14-22](#_ENREF_14)], like the resonant between electron and electrostatic field or electromagnetic wave under magnetic field [[23](#_ENREF_23), [24](#_ENREF_24)].Few studies have incorporated the static electric field into the interactions between electron and E.M wave under magnetic field, despite its relevance in phenomena such as runaway electron generation in tokamaks[[25](#_ENREF_25)] and pitch angle scattering in earth space[[11](#_ENREF_11)] [[26](#_ENREF_26)], where there exist both the electrostatic field, background magnetic field and E.M wave likes whistle wave[[11](#_ENREF_11)]; Building upon pseudopotential approaches used to study electron pitch-angle scattering in E.M waves[[14](#_ENREF_14)], we extend existing models by introducing a static electric field, thereby uncovering a previously unexplored form of resonant trapping in E.M wave–particle interactions. To the best of our knowledge, in this study we first numerically investigate trapping phenomenon in which electron’s velocity within a static electric field become confined at the Normal Doppler Resonant or Anomalous Doppler Resonant in a uniform magnetic field. During trapping, the work performed by the static electric field is continuously converted into gyrokinetic energy or E.M wave, thereby sustaining resonance condition with the E.M wave, and the energy transfer from static electric field also agree with the quantum theory prediction and satisfied with angular conservation model[[27](#_ENREF_27)].

The structure of this paper is as follows: In Section II, we develop a mathematical framework describing the interaction between an electromagnetic wave and an electron subjected to a static electric field that is colinear with a uniform background magnetic field. Section III presents a numerical investigation of the resulting trapping dynamics, including a detailed analysis of the underlying physical mechanism and the derivation of critical threshold conditions necessary for trapping. In Section IV, the theoretical predictions are benchmarked against results from quantum theory to assess consistency and validate the model. Section V offers a comprehensive discussion of the findings, and Section VI concludes the paper with a summary of the principal results and their potential implications.

**II. electron-electromagnetic wave interaction analysis**

**2.1 Field equations**

To analyze interaction between electron and E.M wave, we consider an E.M wave propagate along uniform magnetic field whose phase velocity , where ω is the angular frequency and k is the wavenumber. The uniform background magnetic field is B₀ = B₀ẑ. The wave's magnetic field perturbation B̃ is characterized by the dimensionless parameter ≡ |B̃|/B₀, such that the total magnetic field becomes **B** = B₀ẑ + B̃. The system includes a static electric field E₀ = E₀ẑ, with the total electric field given by **E** = E₀ẑ + Ẽ, where Ẽ represents the electric field components of E.M wave:

Here, is the electric field amplitude of E.M wave, and the polarization satisfies the left-hand circular polarization (LCP) condition when = 1, and right-hand circular polarization (RCP) condition when = -1.

Faraday’s law requires the associated magnetic field to be

Here .

**2.2 Transformation to the wave frame**

In wave frame, which denotes as prime and moves at constant velocity with respect to the lab frame, the fields are

Where . Substituting the wave fields Eq. , Eq. into Eq., Eq. gives and .

Since and {**x**, ict} are relativistic four-vectors, we have

Here and are the wavenumber and position in the wave frame. The magnetic field of E.M wave is

The motion equation of the charge particle in the prime frame is

Where , m is the rest mass of electron and q is the electron charge with q = -e. Note that ,, and differ and should not be confused with each other. The derivation of energy to t’ and the motion equation in each direction should be

Here is the nonrelativistic electron cyclotron frequency in the lab frame with . Introduce

Here And

where , according to Eq. , Eq. , we have

**2.3 Construction of pseudo-potential problem**

Taking the derivative of Eq. with respect to t’ gives

And

According to Eq. and , we have

While in Eq. , consider typically and which is suitable in most case of Tokamak environment, we have

Finally, in the right side of Eq.

While for and , which is satisfied in most of situation, we can confidently ignore the second term in the right side of Eq. , then

Combing Eq. with Eq. , we have

Substituting Eq. into Eq. gives

The time derivation of wave magnetic field in wave frame is

where . By substituting Eq. into Eq. and simplify Eq. gives

Since , finally

**2.3.1 connect with**

To obtain the relationship between and , taking the time derivation of gives

where

and

Here we ignore the time derivation of in Eq. and . Finally, we have

The equation is first-order linear differential equation with the form:

Where , and The exact solution of z is

here and , . Since P<< 1 in most case, we have

Here we choose the initial condition , which implies C0 = 0. As a result, the expression Eq. can simplify to:

The substitution of Eq. into Eq. gives

Integrating Eq. with gives:

Here , and represent the initial condition of , and .

Noting that t = 0 corresponds to z’=0, since z = 0 at t= 0, and recalling the four vectors {}, we have the relation:

Since is in the y direction when z’ = 0 as shown in Eq. , it follows that

Where is defined by and , . With these definitions, Eq. becomes

The substitution of Eq. into Eq. gives:

Simplifying Eq. and substituting for gives:

Here

Normalization of time with we have

The Eq. could also be written as

Where

Multiplying Eq. by and integrating gives a pseudo-energy equation

Where

And

Here .The pseudo potential can only be solved numerically, as it does not have a regular form involving only the parameter , here also depends on t’, which is related to and .

**2.4 Initial condition**

From Eq. , we see that

Here, we use the relation and . At the initial time = 0 and position z = 0, we have

Where

**2.6 Solve**

According to Eq., multiplying on both sides gives

Reorganize the equation, we have

According to Eq., multiplying on both sides gives

Add Eq. and Eq. , we have

Which means energy change ratio equal to work done by static electric field in moving frame.

Using the normalized time , we have

Considering that is mainly determined by , integrating both side with t’ gives

**III. Numerical study of the trapping effect**

According to Eq. , Eq. and Eq., the velocity could be numerically solved by ode45, and can be determined from Eq.. Then, using the four-vector {}, we have:

Finally, all the velocities in lab frame can be solved numerically

3.1 Trapping in anomalous doppler resonance (g = 1)

Quantum analysis indicates that, for anomalous Doppler resonance with electrons, the wave polarization is primarily dictated by the LCP component[[27](#_ENREF_27)], whereas normal Doppler resonance corresponds to the RCP component. We begin by considering two cases in which a LCP electromagnetic wave interacts with electrons in the presence of a uniform magnetic field and a static electric field.

In case I, considering a scenario where the uniform magnetic field B0​ = 2 T and the static electric field E0 = −0.2 V/m, both along the z-axis, which are close to typical Tokamak plasma conditions. A plane LCP E.M wave is assumed to propagate along z direction with refractive index and . The electric field of E.M wave is set to . The numerical results are shown in Figure 1, For the static electric field E0 < 0, the direction of the electric field is opposite to that of the background magnetic field. As a result, the parallel velocity increases over time, as shown in Figure.1(a). Since here , which is given in Eq., the value of decreases as increase according to Eq.. When the parallel velocity satisfies the anomalous doppler resonance condition where , the perpendicular velocity increase abruptly, as shown in Figure.1 (b-d), After the parallel velocity exceeds the resonance condition, the will no longer increase. The phase evolution of and is shown in fig.(e). It can be observed that the fluctuation of is stronger for than for . This is because the change of is proportional to as shown in Eq. . Fig 1.(f) illustrates the pseudo-potential , defined as .The initial pseudo-kinetic energy , which is indicated as red dash line in Fig.1 (f). Although the pseudo-potential increases after the resonance point (ξ<0), the initial pseudo-kinetic energy remains greater than the pseudo-potential throughout, indicating that no trapping occurs.

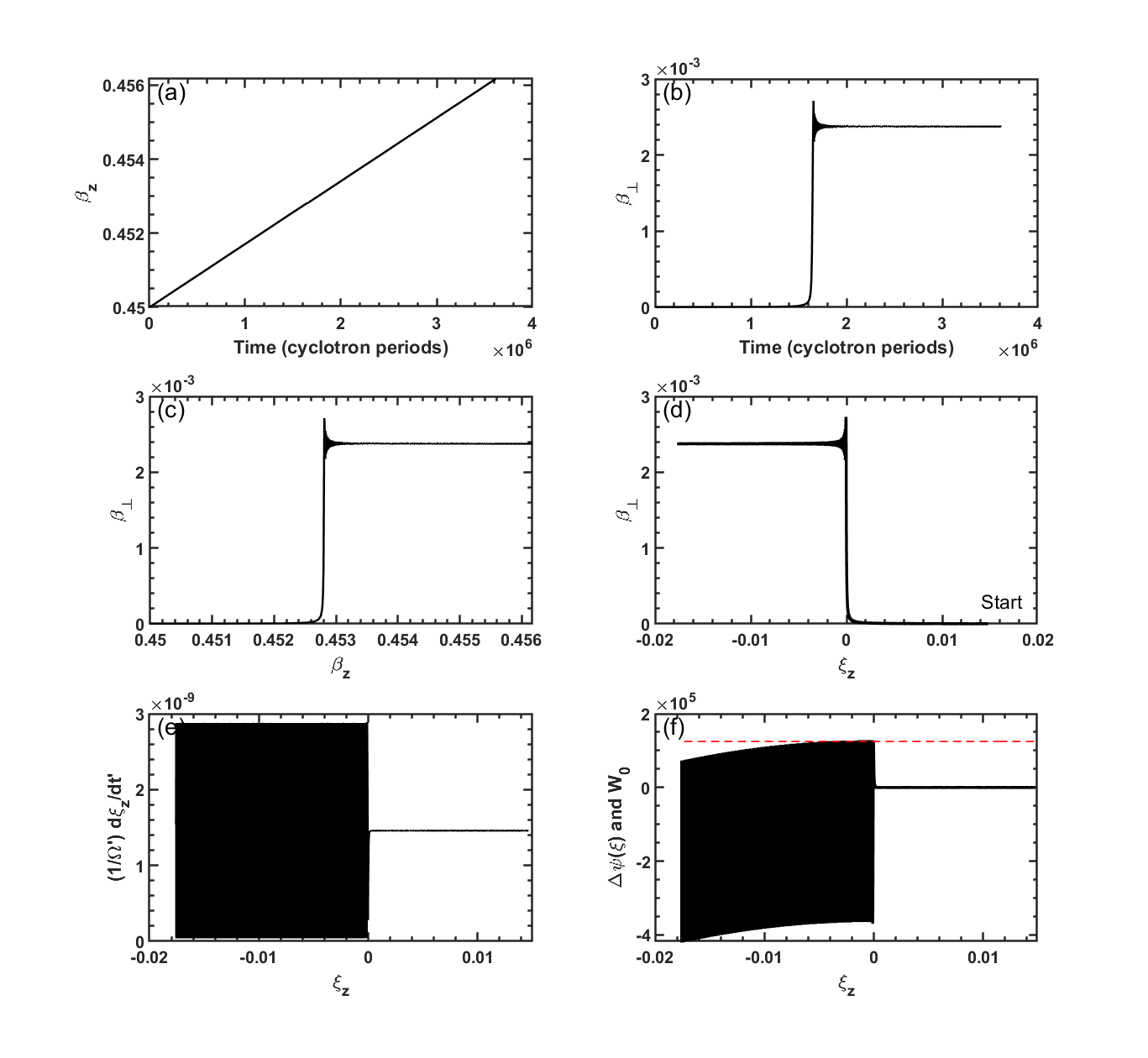


Figure . Numerical integration of Eq. with initial equation Eq.. Input parameters are E0 =- 0.2 V/m, Ew = 20 V/m, B0 = 2 T, ,g = 1, initial = 0.45 and , 0, n = 4. This give (a) The time evolution of . (b) The time evolution of .(c)The velocity phase in ( , . (d)The evolution of with . (e) The evolution of with ξz. (f)The pseudo-potential (black line) and the initial pseudo-kinetic energy E0 (red dash line)

In case II, the electric field of the LCP E.M wave is increased to 40 V/m. As the electron’s parallel velocity approaches the resonant velocity, it no longer increases continuously but begins to oscillate around the resonant velocity, as shown in Fig. 2(a). While on the other hand, the perpendicular velocity increases continuously when trapping in resonant region, as shown in Fig.2 (b-d). The phase trajectory of ( is shown in Fig.2 (e). The closed-loop structure indicates periodic motion around the resonant point, and the direction of motion is labeled with arrow. The electron can only propagate within the region where the pseudo-potential is lower than the initial pseudo-kinetic energy W0. When the pseudo-potential tends to surpass the W0, the electron velocity rebounds upon reaching the boundary of the pseudo-potential well. Consequently, the electron becomes confined within the well, the width of the pseudo-potential well also increases, since it is influenced by the parameter . This bounce effect, shown in Fig.2 (f), illustrates the trapping phenomenon.

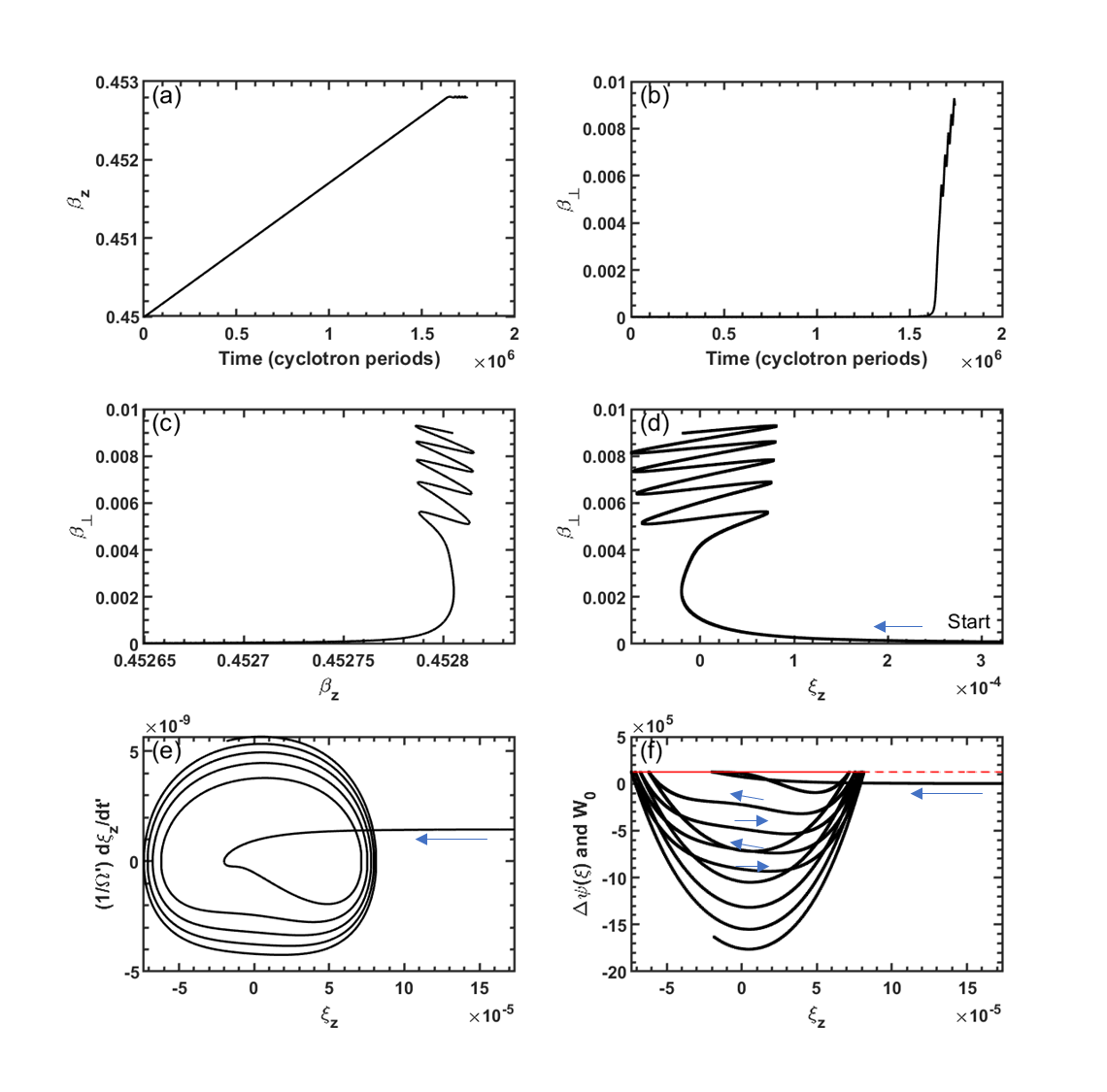


Figure . Same as Fig. (1) except with EW = 22 V/m. This gives but same . Panels (e) and (f) show a zoomed-in view around ξz=0.

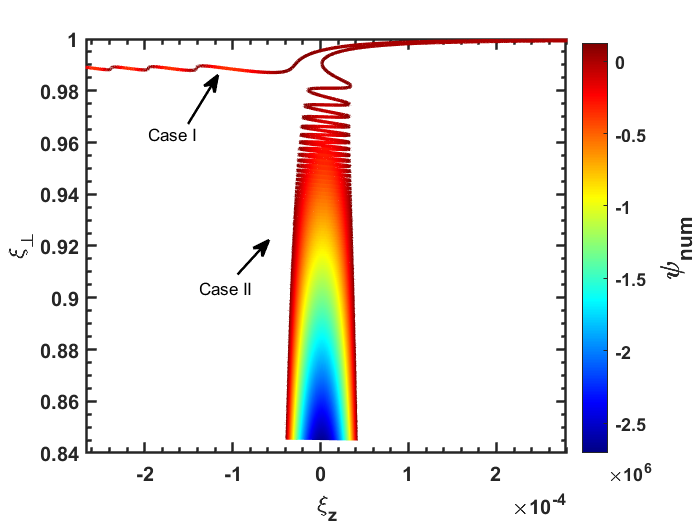


Figure . Pseudo-potential Δψ mapped along the electron trajectory in the () plane.

Since the pseudo-potential Δψ(ξ) is a function of both and ​, we traced Δψ along the particle trajectory in the () phase space. This approach highlights the underlying structure of the pseudo-potential and enables clearer physical interpretation. As shown in Fig. 3, which compares the two scenarios described in Case I and Case II. It can be seen that when the electron becomes trapped under the resonant condition, it slips into a “deep potential valley” that extends further along the direction. In contrast, if the electron passes through the resonant region without being trapped, it continues on a “highway”-like trajectory without further obstruction.

To determine the critical boundary of the trapping region, we refer to Eq. and Eq.. Assuming the initial perpendicular velocity is approximately zero ( ), then s =0. Under this condition, the dynamics are governed solely by two coefficients: and . As shown in Fig. 4, the trapping region is indicated in yellow, while the blue region corresponds to the passing regime. Case I and Case II are marked with star symbols in the figure, located in the passing and trapping regions, respectively. For special case as given in Fig.1, where the critical threshold is , and Consequently, effective electron trapping in the electromagnetic wave requires the LCP electric field intensity to exceed the background static electric field by a factor of more than 102 in typical tokamak plasma with frequencies near the upper-hybrid mode, which is agree with the result in “Constraining Electron Energy in static electric Field via the Anomalous Doppler Resonant with External Electromagnetic Waves”.

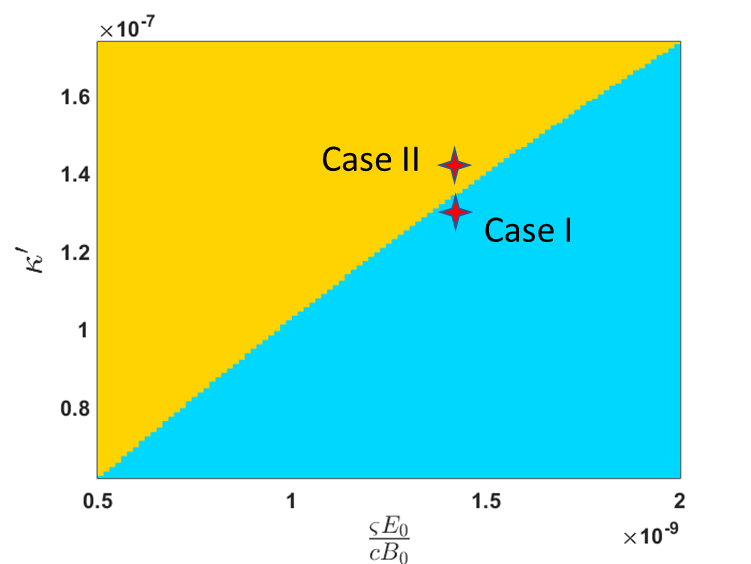


Figure . Parameter space of () showing electron trapping (yellow) and passing (blue) regimes.

3.2 Trapping in normal doppler resonance (g = -1)

According to the angular momentum conservation model[[27](#_ENREF_27)], resonance with electrons under normal Doppler resonant conditions requires that the plane electromagnetic wave possess a right-hand circularly polarized (RCP) component. To illustrate the trapping effect under the normal Doppler resonance, we consider the case where the uniform static electric field E0​ is aligned with the uniform background magnetic field B0B\_0B0​. A plane right-hand circularly polarized (RCP) electromagnetic wave propagates along B0B\_0B0​. Since the electron is accelerated in the negative B0 direction, once resonance with the electromagnetic wave is established, the wave transfers momentum to the electron in the opposite direction of its motion. This counteracts the acceleration, thereby enabling electron trapping under resonant conditions.

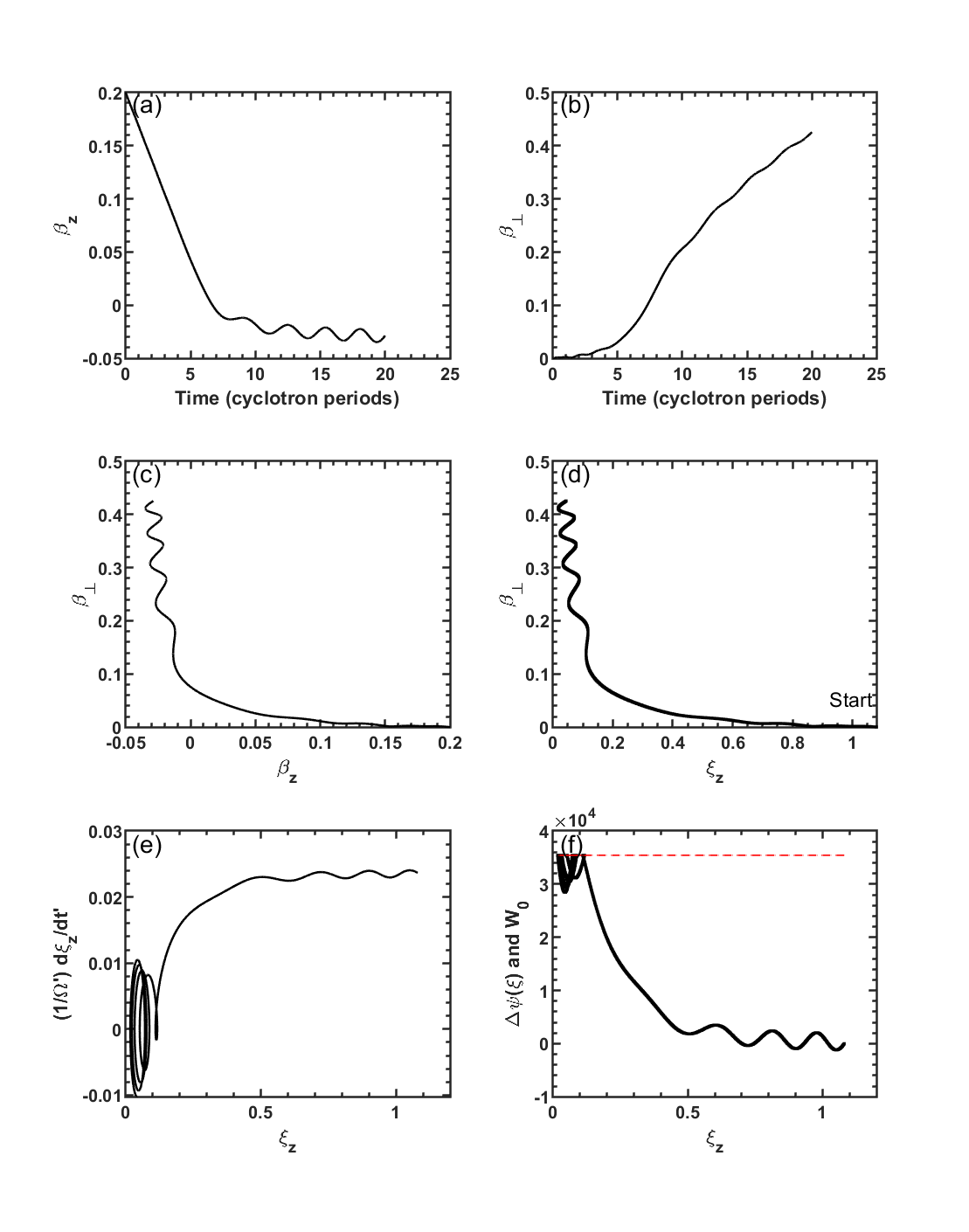
To establish parameters consistent with observations from the Magnetospheric Multiscale[[11](#_ENREF_11)] (MMS) mission, we consider Time Domain Structures (TDS) propagating at a velocity of VTDS/c∼ -0.0333 antiparallel to the local quasistatic magnetic field. Within the TDS, the electric field exhibits an amplitude of E0∼100 mV/m, with one half aligned parallel and the other half antiparallel to the ambient magnetic field. The background magnetic field intensity is B0∼64 nT, and the electron density is ne∼0.15 cm. In the following analysis, we examine the case in which a whistler-mode wave propagates along the magnetic field such that the corresponding normal Doppler resonance velocity coincides with VTDS​, so the electron also be possibility trapped in the TDS structure. Here we consider the initial electron velocity and , furthermore the electron always stays in the TDS with static electric field parallel to the background magnetic field. The cold plasma dispersion relation shows that an electromagnetic wave with ω/Ω = −0.845, refractive index n = 5.4655 and have an NDE resonant speed identical to that of the TDS.

Figure . Same as Fig. (1) but input parameters are E0 = 100 mV/m, Ew = 0.3 V/m, B0 = 64 nT, ω/Ω = -0.845, g = -1, initial = 0.2 and =0, ϕ0= 0, n = 5.4655. This give κ'≈8.254010-2,γT =1.0172, α=-4.5411,γ0=1.0206 and γ’ = 1.0002

The numerical results are shown in Fig. 5, the electron is accelerated in the opposite direction as shown in Fig.5 (a) and get trapped at . As trapping starts, the immediately increase afterward as shown in Fig. 5(b). The phase () is shown in Fig. 5(c). The phase () is shown in Fig. 5(d), Resonance occurs at , where exhibits a pronounced increase. The phase trajectory of ( is shown in Fig.5 (e), and the pseudo-potential is shown in Fig.5 (f). Both exhibit the same structure as in Fig. 2.

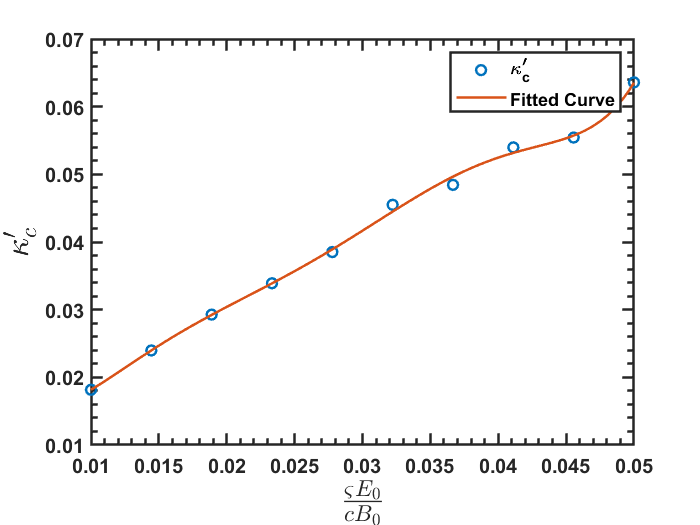


Figure . Critical magnetic field intensity of an E.M wave required to trap electrons under normal Doppler resonance.

Fig. 6 illustrates the critical ratio for trapping under the normal Doppler effect. In the case shown in Fig. 5, the magnetic field ratio of E.M wave is at , while the critical ratio is . Since , the wave amplitude exceeds the threshold required for trapping, allowing electrons to be captured by the wave, as observed in Fig. 5.

**IV. Benchmark with quantum theory**

One characteristic worth pointing out is that when the electron is trapped in the electromagnetic wave, the energy transfer from the static electric field to the gyrokinetic energy is governed by quantum theory (QE theory) [[28](#_ENREF_28)]

Here refers to the Lorentz factor during the resonance. For anomalous Doppler resonance, n = 1 and , can be written as

For Case II, and we have 0.448. To numerically calculate the energy transfer ratio, we evaluate the work done by the static electric field during resonance:

Here refers to the beginning of trapping time. The increase of perpendicular energy is given by

Finally, the energy transfer ratio is calculated as .

The ηₚ values obtained from the two methods are illustrated in Fig. 7. As the electron is trapped by electromagnetic wave, the energy transfer ratio from the numerical results tends to approach the theoretical prediction, and this ratio is independent of the wave’s intensity. The agreement between the numerical and theoretical results confirms both the accuracy of the simulation and the consistency between quantum theory and classical dynamics.

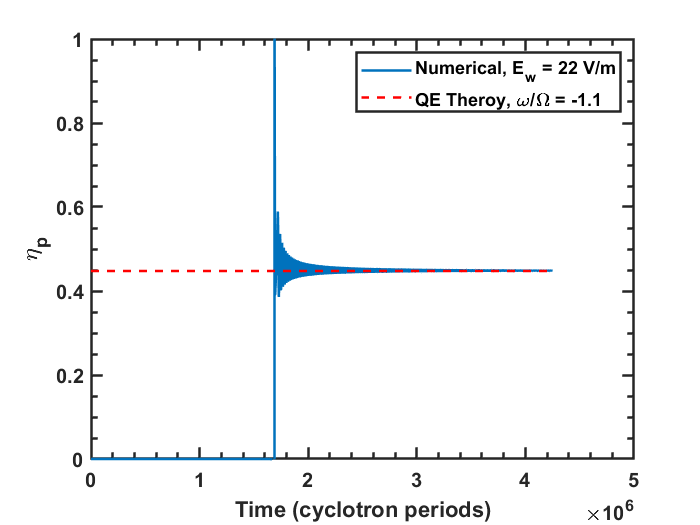


Figure .Energy transfer ratio from the static electric field to the gyrokinetic energy. The parameters used here are the same as in Figure 2 (QE theory means theory based on quantum equation).

**V. Discussion**

This trapping effect can also be qualitatively understood through the conservation of angular momentum and linear momentum. We can draw an analogy by treating the cyclotron electron as a system that contains both internal and kinetic energy, where the internal energy refers to gyrokinetic energy and kinetic energy refers to its translational motion along the magnetic field. When this system is stimulated by an external E.M wave, it undergoes stimulated emission (absorption), radiating (absorbing) E.M wave identical to the external one.

For anomalous doppler resonance, Since the emitted E.M wave propagates in the same direction as the system, conservation of linear momentum requires the system to lose some of its parallel momentum. On the other hand, because the electron possesses right-hand circularly polarization (associated with positive angular momentum), while the emitted wave has left-hand circularly polarization (associated with negative angular momentum), the system must gain angular momentum to conserve the total angular momentum after emission. As a result, the system loses kinetic energy and gains gyrokinetic energy. However, a static electric field continues to replenish the lost kinetic energy. When the rate of energy loss to the wave balances the energy input from the electric field, the electron's parallel velocity ceases to increase, and the electron becomes trapped in the electromagnetic wave. This manifests as a continuous transfer of energy from the static electric field to the gyrokinetic energy of the system.

This explanation could also used for normal doppler

**VI. Summary**

In conclusion, trapping under both normal and anomalous Doppler resonances is analyzed via the pseudo-potential approach. The parallel velocity oscillates within a potential well, while the perpendicular velocity grows continuously. Critical trapping energy is obtained numerically, with energy conservation ratios from simulations and quantum theory showing strong agreement. The mechanism is explained through angular and linear momentum conservation, and potential applications and phenomena are discussed.

Appendix:

**a. Prove the relationship between resonant condition and**

The parameter characterizes the frequency mismatch relative to the resonance condition given by

This relationship can be derived as follows: Starting from the definition of as shown in Eq., we have:

Since {} are four-vector, we have Substituting Eq. and Eq. into Eq. gives

Here represents the resonant condition.

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